4380. *Proposed by George Apostolopoulos.*

Let a, b and c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that

$$a^2 \tan\frac{A}{2} + b^2 \tan\frac{B}{2} + c^2 \tan\frac{C}{2} \le \frac{3\sqrt{3}R^3(R-r)}{2r^2}.$$

We received 11 submissions, including the one from the proposer, all correct. We present two solutions, the second one of which gives a sharper inequality.

Solution 1, by Kee-Wai Lau.

Let S denote the semiperimeter of triangle ABC. The following identities and inequalities are all well known:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{1}$$

$$\sin A\cos A + \sin B\cos B + \sin C\cos C = \frac{rS}{R^2} \tag{2}$$

$$R \ge 2r \quad \textit{(Euler's Inequality)} \tag{3}$$

$$s \le \frac{3\sqrt{3}R}{2} \tag{4}$$

By (1) we have

$$\begin{aligned} a^2\tan\frac{A}{2} &= 4R^2(\sin^2 A)(\tan\frac{A}{2}) \\ &= 4R^2(\sin A)(2\sin\frac{A}{2}\cos\frac{A}{2})(\tan\frac{A}{2}) \\ &= 8R^2(\sin A)(\sin^2\frac{A}{2}) \\ &= 4R^2(\sin A)(1-\cos A). \end{aligned}$$

Similarly, $b^2\tan\frac{B}{2} = 4R^2(\sin B)(1-\cos B)$ and $c^2\tan\frac{C}{2} = 4R^2(\sin C)(1-\cos C)$. Hence, by (1), (2), (3), and (4) we have

$$\begin{aligned} \sum_{cyc} a^2\tan\frac{A}{2} = 4R^2\Big(\sum_{cyc}\sin A - \sum_{cyc}\sin A\cos A\Big) &= 4R^2\Big(\frac{a+b+c}{2R} - \frac{rs}{R^2}\Big) \\ &= 4s(R-r) \\ &\le 6\sqrt{3}R(R-r) \\ &= \frac{3\sqrt{3}R(2r)^2(R-r)}{2r^2} \\ &\le \frac{3\sqrt{3}R^3(R-r)}{2r^2} \end{aligned}$$

and we are done.

Solution 2, by Arkady Alt.

We prove the inequality that

$$\sum_{cyc} a^2 \tan \frac{A}{2} \le 6\sqrt{3}R(R-r)$$

which is sharper than the proposed result since

$$6\sqrt{3}R(R-r) \le \frac{3\sqrt{3}R^3(R-r)}{2r^2} \iff 2r \le R$$

which is Euler's Inequality.

Using the known results that

$$\tan\frac{A}{2} = \frac{r}{s-a}, \quad \sum_{cyc} \frac{a}{s-a} = \frac{4R-2r}{r} \quad \text{and} \quad s \le \frac{3\sqrt{3}R}{2},$$

we obtain

$$\begin{aligned}
&\sum_{cyc} a^2 \tan\frac{A}{2} \le 6\sqrt{3}R(R-r) \iff \\
&\sum_{cyc} \frac{a^2}{s-a} \le \frac{6\sqrt{3}R(R-r)}{r} \iff \\
&\sum_{cyc} \left(\frac{a^2}{s-a} + a\right) \le \frac{6\sqrt{3}R(R-r)}{r} + 2s \iff \\
&\sum_{cyc} \left(\frac{sa}{s-a}\right) \le \frac{6\sqrt{3}R(R-r)}{r} + 2s \iff \\
&s \cdot \left(\frac{4R-2r}{r}\right) \le \frac{6\sqrt{3}R(R-r)}{r} + 2s \iff \\
&s(2R-r) \le 3\sqrt{3}R(R-r) + sr \iff \\
&2s(R-r) \le 3\sqrt{3}R(R-r) \iff \\
&s \le \frac{3\sqrt{3}R}{2}
\end{aligned}$$

and the proof is complete.

4381. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with circumcircle Γ_1 and circumcenter O. Suppose the open ray AO intersects Γ_1 at point D and E is the middle point of BC. The perpendicular bisector of BE intersects BD in P and the perpendicular bisector of EC intersects CD in Q. Finally suppose that circle Γ_2 with center P and radius PE intersects the circle Γ_3 with center Q and radius QE in X. Prove that AX is a symmedian in $\triangle\ ABC$.